MS Analysis Seminar 1:
Bump Functions and Partitions of Unity

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1 Introduction

The goal of this handout is to explore how different decompositions of sets can lead to interesting analytic and topologic structures, particularly centering around the idea of "locally finite" decompositions.
Dual to how the improper Riemann integral is defined by taking the limit of integrals on increasing intervals of $\mathbb{R}$, this technology proves helpful in the definition of the extended Riemann integral by breaking up functions into "increasing pieces."

2 Locally Finite Covers

We start with the following theorem, which provides us with an explicit construction for interesting covers of open sets.

Theorem 1. *Let $U = \bigcup_{m \in \mathbb{N}} U_m \subseteq \mathbb{R}^n$ all be open. Then there exists a collection of rectangles $\{R_i\}_{i \in \mathbb{N}}$ such that the following hold:*

- $U \subseteq \bigcup_{i \in \mathbb{N}} R_i^\circ$.
- *For all $i \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $R_i \subseteq U_m$.*
- *For all $x \in U$, there exists $\varepsilon > 0$ with $B(x, \varepsilon)$ intersecting only finitely many R_i.*

Proof. We start with the following lemma, which provides us with useful auxillary sets:

Lemma 1 (Exhaustion by Compact Sets)**.** *Suppose U is open. Then there exists a countable collection $\{C_i\}_{i \in \mathbb{N}}$ of compact sets such that $A = \bigcup_{i \in \mathbb{N}} C_i$ and $C_i \subseteq C_{i+1}^\circ$ for all i.*

Proof. First recall that given any $X \subseteq \mathbb{R}^n$, the **distance function** $\operatorname{dist}(\cdot, X)$ given by

$$\operatorname{dist}(x, X) = \inf_{y \in X} d(x, y)$$

is well-defined and continuous. Now, for each $i \in \mathbb{N}$, we can set

$$C_i = \{x : \operatorname{dist}(x, U^c) \geq 1/i\} \bigcap B[0, i]$$

which is clearly compact as the intersection of a closed set and a compact set.
We now check the two conditions as stated in the lemma. To see that the union of these sets is equal to A, first observe that given any $x \in C_i$,

$$\operatorname{dist}(x, U^c) > 0 \implies x \notin U^c \implies x \in U \implies A \supseteq \bigcup_{i \in \mathbb{N}} C_i$$

and to see the reverse inclusion, observe that since U is open, given any $x \in U$, we may fix $\varepsilon > 0$ with

$$B(x, \varepsilon) \subseteq U \implies \operatorname{dist}(x, U^c) > \varepsilon$$

so we may choose $i \in \mathbb{N}$ such that both $1/i < \varepsilon$ and $i > d(0, x)$ hold to see that $x \in C_i$ as desired.
To see that these C_i nest in the desired fashion, simply note that

$$C_i \subseteq \{x : d(x, U^c) > 1/(i+1)\} \bigcap B(0, i+1) \subseteq C_{i+1}^\circ$$

where the second term is an intersection of open sets and hence open (and contained in C_{i+1}°) as desired. □

Now let C_i be as above, and for convenience put $C_i = \varnothing$ for $i \leq 0$. For each $i \in \mathbb{N}$ we now define $D_i = C_i \setminus C_{i-1}^\circ$. Observe that D_i is bounded (as a subset of C_i) and closed (as $C_i \cap (C_{i+1}^\circ)^c$) and hence compact. Furthermore, D_i is completely disjoint from $C_{i-2} \subseteq C_{i-1}^\circ$. Now for each $x \in D_i$ choose m with $x \in U_m$ and ε with $B(x, \varepsilon) \subseteq U_m \cap C_{i-2}^c$. Then choosing a rectangle $R_x \subseteq B(x, \varepsilon)$ for every such x, we observe that $\{R_x^\circ\}_{x \in D_i}$ is an open cover of D_i. Invoking compactness, we can now pass down to a finite $\{R_j^\circ\}_{j \in [n]}$ covering D_i. To conclude, we then take the union of these (finite) collections over all $i \in \mathbb{N}$ to produce the desired collection $\{R_j\}_{j \in \mathbb{N}}$.
Given this collection, we now check the desired properties. Clearly the first two hold, so we only check the last one.
Let $x \in U$ be arbitrary. Then $x \in C_{i-1} \subseteq C_i^\circ$ for some i and we can choose $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq C_i^\circ$. But by construction, all of C_i° can only have nontrivial intersection with the rectangles covering $D_1, \cdots, D_{i+2}$, which is a finite collection, so we're done. □

3 Bump Functions

In order to better "support" (haha) the covers we developed in the previous part, we now quickly prove a couple results about the existence of C^∞ "indicator functions" that'll be useful in our partitions of unity.

Theorem 2. *Let $R = \prod_{i \in n} [a_i, b_i]$. Then there exists $I_R \in C^\infty(\mathbb{R}^n; \mathbb{R})$ such that $I_R(x) > 0$ for all $x \in R^\circ$ and 0 otherwise.*

Proof. We'll explicitly construct such a function in $\mathbb{R}$, then use some coordinate trickery to construct our desired function.

Claim.

$$g : \mathbb{R} \to \mathbb{R} \text{ via } g(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & \textit{otherwise} \end{cases}$$

is C^∞.

Proof. We'll proceed by showing that for each $n \in \mathbb{N}$ that

$$g_n(x) = \begin{cases} e^{-1/x}/x^n & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is continuous at 0. To do so, first recall that that $\alpha < e^\alpha$ for all $\alpha \in \mathbb{R}$. Setting $\alpha = t/2n$, we find that

$$t/2n < e^{t/2n} \implies \frac{t^n}{e^t} < \frac{(2n)^n}{e^{t/2}}$$

which implies

$$e^{-1/x}/x^n < (2n)^n \cdot e^{-1/x}$$

which goes to 0 as desired. To conclude, observe that g and all of it's nth derivatives are given precisely as linear combinations of g_ns as defined above (in particular noticing that the derivative of g_n at 0 is given exactly by the limit as $x \to 0$ of g_{n+1}) to see that g is C^∞ as desired.

$\square$

Now observe that we can manipulate g to produce a C^∞ function $f = g(x) \cdot g(1-x)$ that's positive on $(0, 1)$ and 0 everywhere else. Now we can let

$$I_R(x) = \prod_{i \in [n]} f\left(\frac{x_i - a_i}{b_i - a_i}\right)$$

and conclude the proof. $\square$

To better support the content of the next section, for $f : \mathbb{R}^n \to \mathbb{R}$ we'll define the **support** S_f of f to be

$$S_f = \overline{\{f \neq 0\}}$$

4 Partitions of Unity

Now that all of our technology has been assembled, we'll finish with the following result:

Theorem 3. *Let $U = \bigcup_{k \in \mathbb{N}} U_k \subseteq \mathbb{R}^n$ all be open. There there exists $\{f_i\}_{i \in \mathbb{N}} \subseteq C^\infty(\mathbb{R}^n; \mathbb{R})$ such that:*

- *f_i is non-negative, and the support of each f_i is compact and contained in U.*
- *For all $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon)$ intersects only finitely many S_{f_i}.*

- $\sum_{i \in \mathbb{N}} f_i = x \mapsto \begin{cases} 1 & x \in U \\ 0 & \textit{otherwise} \end{cases}$
- *For all* $i \in \mathbb{N}$*, there exists* $k \in \mathbb{N}$ *such that* U_k *contains the support of* f_i*.*

We call such a collection a partition of unity of class C^∞ *with compact support dominated by* $\{U_k\}_{k \in \mathbb{N}}$*.*

Proof. Let $\{R_i\}_{i \in \mathbb{N}}$ be as in the first theorem, and $\{I_i\}_{i \in \mathbb{N}}$ be the corresponding indicator functions from the second result. Let $\lambda = \sum_{i \in \mathbb{N}} I_i$, which, by our local finiteness condition, clearly converges and is C^∞. Now set

$$f_i = \begin{cases} I_i/\lambda & x \in U \\ 0 & \text{otherwise} \end{cases}$$

to produce the desired functions. $\square$