Area Exam stuff

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Contents

1 About

The following is a set of notes I took for my area exam at Stanford. Since it was intended as a way for me to last-minute review things, everything is written up rather tersely, in particular the section on wave equations. Regardless, I've yet to find a publicly accessible and somewhat-cleanly-written-up version of the result in 3.3, so hopefully this is a useful reference for that (or for someone trying to crash-course the other topics). As mentioned in the text, the section on wave equations roughly follows the Holzegel/Luk notes. The section on fluids roughly follows the Bedrossian-Vicol book, as well as a section of the Majda-Bertozzi book, and the section on harmonic analysis follows Ch 7-9 of Muscalu-Schlag; see the syllabus posted on my site for a more precise citation.

2 Talk abstract

Beginning with the celebrated work of Christodoulou-Klainerman in 1993, the stability of Minkowski space and other special solutions to the Einstein equations has remained a central focus of mathematical GR. Motivated by their work as well as further refinements by Lindblad-Rodnianski, Shen, Keir, and many others, we present a general method for proving small data global existence for a class of quasilinear wave equations, which includes the Einstein vacuum equations in harmonic gauge. In addition to allowing for a wider class of initial data than previous results, our method requires commuting with a smaller set of vector fields than needed for e.g. Klainerman's Sobolev inequality, which allows us to prove decay using an elliptic estimate involving the wave operator, as opposed to e.g. the r^p method of Dafermos-Rodnianski.

3 Wave Equations and General Relativity

3.1 Local existence and uniqueness for wave equations

Local existence done via Picard iteration and energy estimates. Uniqueness follows from energy estimates.

3.2 Dispersive estimates for linear wave equations

Can be done with exact formulas for the fundamental solution. Can also be done with the following functional estimate:

Theorem 1 (Klainerman-Sobolev). *Let Γ be the full set of commuting vector fields. For any sufficiently smooth ϕ, we have the pointwise bounds*

$$|\partial\phi(u,v,\theta)| \lesssim u^{-1/2}v^{-1}\sum_{|\alpha|\leq 4}\|\Gamma^\alpha\partial\phi\|_{L^2}.$$

Furthermore,

$$\left|\overline{\partial}\phi(u,v,\theta)\right| \lesssim v^{-3/2}\sum_{|\alpha|\leq 4}\|\Gamma^\alpha\partial\phi\|_{L^2}.$$

3.3 Blowup for $\Box u = (\partial_t u)^2$

The goal of this section is to show that all global solutions to the equation

$$\begin{cases}\Box u = (\partial_t u)^2\\ u(t=0) = u_0\\ \partial_t u(t=0) = u_1\end{cases} \tag{1}$$

with u_i smooth and compactly supported are trivial, implying that all nontrivial solutions blowup in finite time. Following the Holzegel/Luk notes, we will deduce this via a reduction to spherical means and an ODE blowup type result.

3.3.1 Preliminaries

We begin with the Darboux equation. For $h \in C^\infty(\mathbb{R}^n)$, define

$$M_h(x,r) := \frac{1}{|B(x,r)|}\int_{B(x,r)} h(y)dy = \int_{\mathbb{S}^1} h(x+rz)dz.$$

We claim the following:

Theorem 2. *With M_h defined as above, we have*

$$\Delta_x M_h(x,r) = \left(\partial_r^2 + \frac{n-1}{r}\partial_r\right) M_h(x,r).$$

Proof. By definition, we have

$$|B(0,1)| \int_0^R r^{n-1} M_h(x,r) dr = \int_{|y|\le R} h(x+y) dy.$$

Taking Δ_x on both sides and integrating by parts, we deduce that

$$|B(0,1)| \int_0^R r^{n-1} \Delta_x M_h(x,r) dr = \int_{|y|\le R} \Delta_x h(x+y) dy = \int_{|y|\le R} \partial^i \partial_i h(x+y) dy = \int_{|y|=R} \frac{y^i}{R} \partial_i h(x+y) dy.$$

Changing variables to $z = y/R$, this is further equal to

$$R^{n-1} \int_{\mathbb{S}^1} z^i \partial_i h(x+rz) dy = |B(0,1)|\, R^{n-1} \partial_r M_h(x,r).$$

Now taking derivatives with respect to r, we deduce that

$$R^{n-1} \Delta_x M_h(x,r) = (n-1) R^{n-2} \partial_r M_h(x,r) + R^{n-1} \partial_r^2 M_h(x,r)$$

as desired. □

We will also need the following calculation, where all functions are now living in $\mathbb{R}^{n+1}$:

Lemma 3. *If $\Box u = F$, then*

$$M_F(0,r) = -\partial_t^2 M_u(0,r) + \left(\partial_r^2 + \frac{n-1}{r}\partial_r\right) M_u(0,r).$$

where now M_F implicitly also may depend on time.

Proof. For any fixed r, we have

$$(-\partial_t^2 + \Delta_x) M_u(x,r) = \Box_x M_u(x,r) = M_{\Box u}(x,r)$$

so using the previous equation and plugging in $x = 0$ yields the result. □

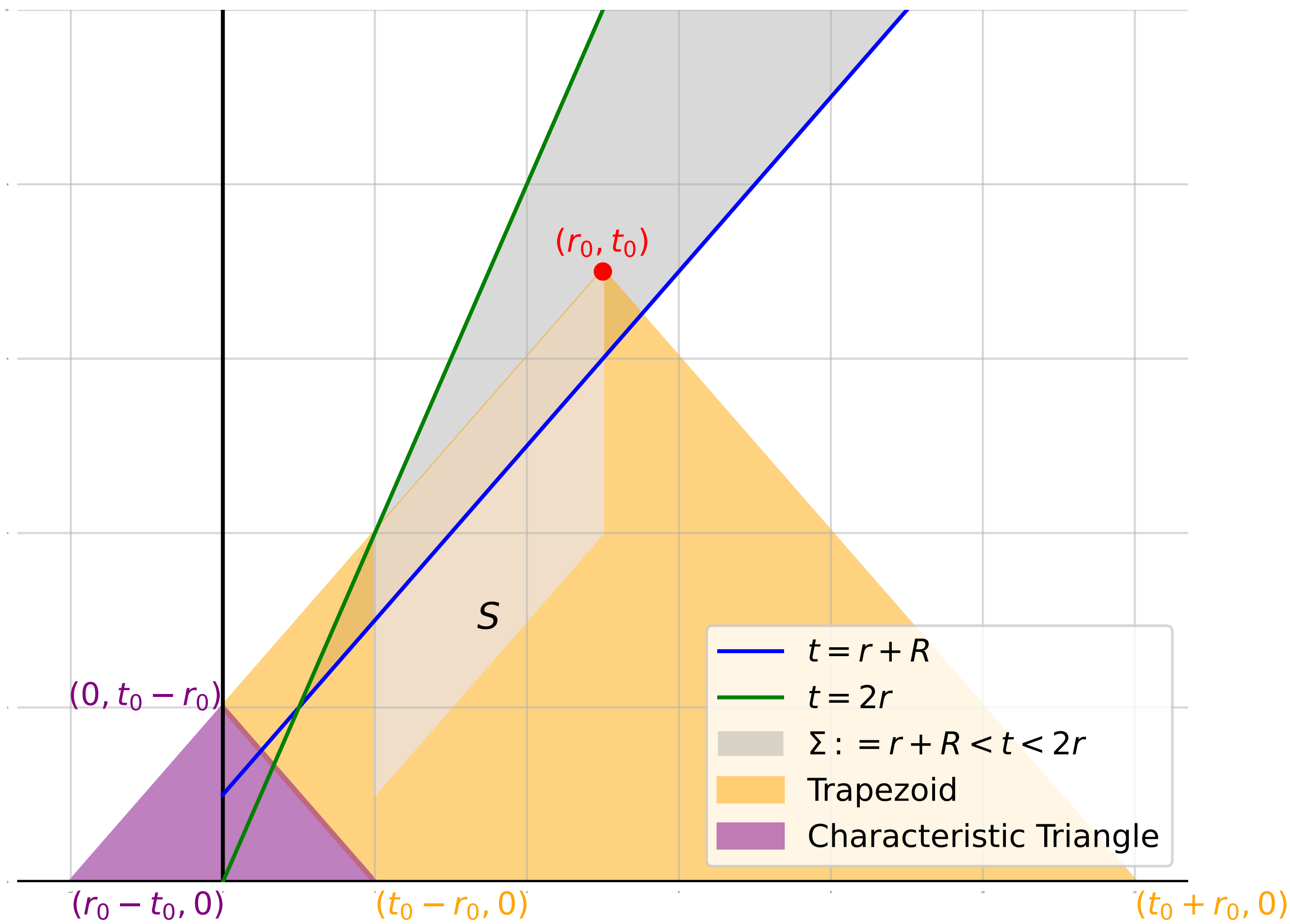
Finally, we will need the following explicit formula for solutions to the wave equation in $1+1$ dimensions.

Theorem 4. *The solution to the equation $\Box v = F$ with initial data $v(t=0) = v_0$ and $\partial_t v(t=0) = v_1$ is given by*

$$v(t,r) = \frac{1}{2}\left[v_0(t-r) + v_0(t+r) + \int_{|r-r'|\le t} v_1(r') dr' + \int_{T(t,r)} F(t',r') dt' r'\right]$$

where $T(t,r) := \{(t',r') \mid t' \le t, |r-r'| \le t-t'\}$ is the backward light cone from (t,r).

3.3.2 The Proof



Now suppose we have a global C^2 solution of (1), and take R to be such that the initial data is supported inside of $B(x, R)$. Define $v(t, r) := M_u(0, r)$ and $u := t - r$. Note that $\partial_r^2(rv) = r(\partial_r^2 + \frac{2}{r}\partial_r)v$, so using lemma 3, we know that rv satisfies the $1 + 1$ dimensional wave equation

$$\partial_t^2(rv) - \partial_r^2(rv) = rF =: rM_{(\partial_t u)^2}$$

In particular, using Theorem 4 and dividing by r, we have that

$$v(t_0, r_0) = \frac{1}{2r_0}\left(\tilde{V} + \int_{T(r_0,t_0)} rF drdt\right)$$

where $\tilde{V}$ is a solution to $\square\tilde{V} = 0$ with the correct data. For $(t_0, r_0) \in \Sigma := \{r + R < t < 2r\}$, the contribution from the homogeneous solution vanishes, and hence

$$v(t_0, r_0) = \frac{1}{2r_0}\left(\int_{T(r_0,t_0)}^{t_0+r_0} rF drdt\right) = \frac{1}{2r_0}\left(\int_{T(r_0,t_0)-T(0,u_0)} rF drdt\right) \geq \frac{1}{2r_0}\left(\int_{T^*(r_0,t_0)} r(\partial_t v)^2 drdt\right) \quad (2)$$

where the last inequality follows from Jensen's inequality. By positivity, we can further restrict the area of integration on the right hand side to the set

$$\{u_0 < r < r_0, -R < u < u_0\}$$

to replace the right hand side by

$$\frac{1}{2r_0}\int_{u_0}^{r_0} rdr \int_{r-R}^{r+u_0} (\partial_t v)^2 dt.$$

Now note that

$$|v(r, r+u_0)| = \left|\int_{r-R}^{r+u_0} \partial_t v(r,t)dt\right| \le (u_0+R)^{1/2}\left|\int (\partial_t v)^2\right|^{1/2}$$

so plugging this into the previous equation yields

$$v(t_0, r_0) \ge \frac{1}{2r_0}\int_{u_0}^{r_0} rdr \int_{r-R}^{r+u_0} (\partial_t v)^2 dt \ge \frac{1}{2r_0(u_0+R)}\int_{u_0}^{r_0} rv(r, r+u_0)dr$$

Now define

$$\beta(r_0) := \int_{u_0}^{r_0} rv(r, r+u_0)^2 dr$$

and note

$$\beta'(r_0) = r_0 v(r_0, r+u_0)^2 \ge \frac{1}{4(R+u_0)r_0}\beta^2$$

by the equation above. Integrating this functional inequality implies that, if $\beta(r_0) \neq 0$, then

$$\frac{1}{\beta(r_0)} \ge \frac{1}{\beta(r_0)} - \frac{1}{\beta(r)} \ge \frac{1}{4}\frac{1}{(R+u_0)^2}\log\frac{r}{r_0}$$

for all r, which is impossible. We conclude that $\beta = 0$ in Σ, hence $v = 0$ in Σ. Now using eq. (2), we deduce that $v \equiv 0$ on a full slice, which concludes.

3.4 Null condition and global existence for semilinear problems with slowly decaying data

Theorem 5. *Small data solutions are global.*

4 Fluid Mechanics

4.1 Basics

Recall: Eulerian and Lagrangian viewpoints. In Eulerian view, the fundamental quantities to study are the velocity field u, the density ρ, and the pressure p.
In the Lagrangian viewpoint, the fundamental quantity is the flow map $X(t,a)$ which satisfies the ODE

$$\partial_t X(t,a) = u(t, X(t,a)).$$

We also have the back-to-labels map A which satisfies

$$A(t, X(t,a)) = a.$$

Lemma 6. *We have the estimate*

$$|\nabla_a X(t,a)| \le \exp\left(\int \|\nabla u\|_{L^\infty}\right)$$

Proof. We first compute that

$$\partial_t\partial_k X^j(t,a) = \partial_k\partial_t X^j(t,a) = \partial_k u^j(t, X(t,a)) = \partial_l u^j(t, X(t,a))\cdot\partial_k X^l(t,a) \lesssim \|\nabla u\|\left|\partial_k X^l(t,a)\right|$$

so Gronwall gives the result. □

Lemma 7. *The back to labels map satisfies*

$$\partial_t A + u \cdot \nabla A = 0$$

Proof. Taking the derivative of the identity $x = X(t, A(t,x))$ with respect to t yields

$$-u(t,x) = \partial_j X(t, A(t,x))\partial_t A^j(t,x).$$

On the other hand, we also have

$$\delta^i_j = \partial_k A^i(t, X(t,a))\partial_j X^k(t,a)$$

so contracting the first equation with $\partial_k A^i(t, X(t,a))$ gives

$$-u \cdot \nabla A(t, X(t,a)) = \partial_t A^i(t, X(t,a))$$

which implies the result. □

Lemma 8. *We have*

$$\|\nabla A\|_{L^\infty} \lesssim \exp\left(\int \|\nabla u\|_{L^\infty}\right)$$

and hence

$$\left|\log\left(\frac{|a-b|}{|X(t,a)-X(t,b)|}\right)\right| \leq \int \|\nabla u\|_{L^\infty}.$$

Proof. Take a spatial gradient of the equation from the previous lemma, then use chain rule to compute $\partial_t \nabla A(t, X(t,a))$, substituting the new equation as necessary. □

Lemma 9. $X(t,\cdot)$ *induces a volume preserving diffeomorphism iff* $\operatorname{div} u = 0$.

Proof. This reduces to showing that the Jacobian determinant of $\nabla_a X(t,a)$ is constant iff u is divergence free. Recall that, by Jacobi's formula,

$$(\det M)' = \operatorname{tr}(\operatorname{adj} M M').$$

Hence

$$\begin{aligned}
\partial_t \det \nabla X(t,a) &= \operatorname{tr}((\operatorname{adj} \nabla X(t,a))\partial_t \nabla X(t,a)) \\
&= \operatorname{tr}((\operatorname{adj} \nabla X(t,a))\nabla_x u(t, X(t,a))\nabla_a X(t,a)) \\
&= \operatorname{tr}(\nabla_a X(t,a)(\operatorname{adj} \nabla X(t,a))\nabla_x u(t, X(t,a))) \\
&= \det(\nabla X(t,a)) \operatorname{div} u
\end{aligned}$$

which is enough. □

Theorem 10. *Let V be a volume and $V(t)$ be it's pushforward. Then*

$$\partial_t \int_{V(t)} f = \int_{V(t)} \partial_t f + \nabla \cdot (fu)$$

The Euler equations read

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p \\ \nabla \cdot u = 0 \end{cases}$$

Lemma 11. *Given u, one can recover the pressure via taking the divergence:*

$$-\Delta p = \nabla \cdot (u \cdot \nabla u) = \partial^2_{ij}(u^i u^j)$$

so

$$p = (-\Delta)^{-1}\partial^2_{ij}(u^i u^j)$$

which is a Mikhlin multiplier, implying that

$$\|p\|_{L^q} \lesssim \|u\|^2_{L^{2q}}.$$

Vorticity formulation of Euler in 3D:

$$D_t \omega = (\omega \cdot \nabla) u.$$

Note that 2D solutions can be embedded in 3D by simply assuming no independence on the z coordinate; using this correspondence, the RHS vanishes in 2D, since $\omega = (0, 0, \bar{\omega})$ and all z derivatives of u vanish. Also note that the velocity can be recovered given the voriticity via the Biot Savart law: u is given by the curl of a stream function ψ given by $\psi = (-\Delta)^{-1}\omega$, which can equivalently be written on the Fourier side as

$$\hat{u} = \frac{i\xi}{|\xi|^2} \times \hat{\omega}$$

4.2 Local Well Posedness for Euler and Navier Stokes

4.2.1 The energy estimate for Euler and mollified Euler

Theorem 12. *Suppose u is a solution to Euler. Then*

$$\frac{d}{dt} \|u\|_{H^s}^2 \lesssim \|u\|_{H^s} \left\| \widehat{\nabla u} \right\|_{L^1}$$

Proof. We compute directly that, using the fact that $\mathbb{P}u = u$ and that $\mathbb{P}$ is self-adjoint,

$$\frac{d}{dt} \|u\|_{H^s}^2 = -\int \langle \nabla \rangle^s u \cdot (u \cdot \nabla) (\langle \nabla \rangle^s u) - \int \langle \nabla \rangle^s u \cdot [\langle \nabla \rangle^s, u \cdot \nabla] u.$$

The first term vanishes upon integrating by parts. Now we use the expansions

$$\mathcal{F}[\langle \nabla \rangle^s u^\mu \partial_\mu u](\eta) = \int \langle \eta \rangle^s \hat{u}^\mu(\eta - \xi) i\xi^\mu \hat{u}(\xi) d\xi$$

and

$$\mathcal{F}[u^\mu \partial_\mu \langle \nabla \rangle^s u](\eta) = \int \langle \xi \rangle^s \hat{u}^\mu(\eta - \xi) i\xi^\mu \hat{u}(\xi) d\xi$$

to substitute into the second term, which becomes

$$\int \int \langle \eta \rangle^s \hat{u}(\eta) \left(\langle \xi \rangle^s - \langle \eta \rangle^s\right) \hat{u}(\eta - \xi) \cdot i\xi \hat{u}(\xi) d\xi d\eta.$$

Now we use the functional inequality

$$|\langle A \rangle^s - \langle B \rangle^s| \leq |A - B| \left(\langle A \rangle^{s-1} + \langle A - B \rangle^{s-1}\right)$$

and Young's convolution inequality to bound all the convolution type terms in L^2 by

$$\|u\|_{H^s} \|\nabla u\|_{L^1} \lesssim \|u\|_{H^s}^2.$$

□

Theorem 13. *Euler is locally well posed in H^s for $s > d/2 + 1$.*

Proof. Consider the mollified version of the equation:

$$\partial_t u^\varepsilon = J_\varepsilon \mathbb{P}((J_\varepsilon u^\varepsilon \cdot \nabla) J_\varepsilon u^\varepsilon) =: F_\varepsilon(u).$$

We first claim that solutions to this equation exist locally, with the time of existence possibly depending on ε. To show this, we treat this equation as a H^s valued ODE, noting that F_ε is locally Lipschitz with Lipschitz constant $\mathcal{O}(\varepsilon^{-1})$, which allows us to apply Picard's theorem to deduce local existence. Now using the energy estimate (which still holds for the mollified equation) we can Gronwall to show that $\|u^\varepsilon\|_{H^s}$ can

be estimated uniformly in ε, and hence the existence time can be made independent ε.
Furthermore, by the energy estimate, u^ε can be shown to be uniformly bounded in H^s and Lipschitz in H^{s-1}. We can now show that u^ε is L^2 Cauchy. It suffices to show

$$\left\|u^\varepsilon - u^\delta\right\|_{L^2}(t) \lesssim \max(\varepsilon, \delta) \left\|u_0\right\|_{H^s}.$$

To do so, first note that

$$\widehat{(\phi_\varepsilon - \phi_\delta)} = \hat{\phi}(\varepsilon\xi) - \hat{\phi}(\delta\xi) \lesssim |\varepsilon - \delta| \, |\xi|$$

which implies

$$\|(J_\varepsilon - J_\delta)f\|_{H^s} \lesssim |\varepsilon - \delta| \, \|f\|_{H^{s+1}}.$$

Now observe that

$$\partial_t \|u_\varepsilon - u_\delta\|_{L^2}^2 = \left\langle u^\varepsilon - u^\delta, F_\varepsilon(u^\varepsilon) - F_\delta(u^\delta)\right\rangle.$$

Using the mollifier properties and expanding the difference of the second term, everything works out. Now observe by weak compactness, the limit point $u \in \mathrm{Lip}(H^{s-1}) \cap L^\infty(H^s)$, with the sequence u^ε converging weak star. Furthermore, by lower semicontinuity of weak star convergence, u is bounded uniformly in these spaces.
Using the integral formulation of the Euler equations and the convergence above shows the desired result. □

4.3 Continuation Criteria

The above gives us various blowup criteria for the solution; in particular, it shows that the solution can be continued so long as $\|u\|_{H^s}$ remains bounded. We now improve this blowup criteria for initial data in H^s, assuming s is an integer.

Theorem 14. *The solution can be continued as long as $\int \|\nabla u\|_{L^\infty} < \infty$.*

Proof. Since $\|u\|_{L^2}$ is conserved, it suffices to show the estimate

$$\frac{d}{dt}\|u\|_{\dot{H}^s}^2 \lesssim \|u\|_{H^s} \|\nabla u\|_{L^\infty}$$

so that one can Gronwall and deduce that $\|u\|_{\dot{H}^s}$ and hence $\|u\|_{H^s}$ remains bounded. We first recall the following Gagliardo Nirenberg type inequality:

$$\|f\|_{\dot{W}^{i,2m/i}} \lesssim \|f\|_{L^\infty}^{1-i/m} \|f\|_{\dot{H}^m}^{i/m}.$$

Now we compute that

$$\frac{d}{dt}\|\partial^\alpha u\|_{L^2} = -\int \partial^\alpha u \partial^\alpha \mathbb{P}(u\cdot\nabla)u \lesssim -\sum_{\beta<\alpha}\int \partial^\alpha u((\partial^\beta u)\cdot\nabla)\partial^{\alpha-\beta}u$$

(note that the $\beta = \alpha$ term drops out upon using the fact that u is divergence free). Applying the estimate above and checking that everything scales correctly concludes. □

The above shows that L^1L^∞ control of ∇u is enough to continue a solution. It turns out that this can be even further sharpened to only asking for control over the vorticity.

Theorem 15 (Beal-Kato-Majda)**.** *The solution can be continued as long as $\int \|\omega\|_{L^\infty} < \infty$.*

Proof. We first show that SIOs are L^∞ bounded up to a log loss.

Lemma 16. *Suppose T is given by a homogenous SIO satisfying the cancellation condition. Then*

$$|Tf| \lesssim \|f\|_{L^2} + \|f\|_{L^\infty} (1 + \log_+ \frac{[f]_{C^{0,\alpha}}}{\|f\|^\infty}) \lesssim \|f\|_{L^2} + \|f\|_{L^\infty} (1 + \log_+ \frac{\|f\|_{H^s}}{\|f\|^\infty})$$

and if $f = \nabla g$, then

$$|Tf| \lesssim \|g\|_{L^2} + \|f\|_{L^\infty} (1 + \log_+ \frac{\|f\|_{H^s}}{\|f\|^\infty}).$$

Proof. Decompose the principal value into scales $\ll 1, \leq 2$, and ≥ 2. Use Holder regularity to handle the singularity at 0, and optimize in the scale chosen. □

Now recall from the previous part that

$$\partial_t \|u\|_{H^s}^2 \lesssim \|u\|_{H^s}^2 \|\nabla u\|_{L^\infty}.$$

Since ∇u is given by an SIO + a bounded operator on ω, we have

$$\partial_t \|u\|_{H^s}^2 \lesssim \|u\|_{H^s}^2 \left(\|u\|_{L^2} + \|\omega\|_{L^\infty} (1 + \log_+ \frac{\|f\|_{H^{s-1}}}{\|\omega\|_{L^\infty}}) \right) \lesssim \|u\|_{H^s}^2 \left(\|u\|_{L^2} + \|\omega\|_{L^\infty} (1 + \log_+ \frac{\|u\|_{H^s}}{\|\omega\|_{L^\infty}}) \right).$$

Integrating this allows one to Gronwall and deduce boundedness of $\log(1 + \|u\|_{H^s})$. □

Note as a consequence of the above and the fact that $\|\omega\|_{L^\infty}$ is conserved in 2D (as it's simply transported along the flow), we immediately deduce that all solutions to Euler in 2D are global.

4.4 Local existence for NSE

We now consider the problem of constructing solutions to the Navier Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p \\ \nabla \cdot u = 0 \end{cases}$$

for $\nu > 0$. We first remark that a similar energy estimate holds as for Euler; in fact, the extra heat term comes with a good sign, so that the estimate reads

$$\frac{d}{dt} \|u\|_{H^s}^2 + \nu \|\nabla u\|_{H^s} \lesssim \|u\|_{H^s} \left\| \widehat{\nabla u} \right\|_{L^1}.$$

Now we do everything in the same way as Euler, with the mollified analogue of the extra heat term being $J_\varepsilon \Delta J_\varepsilon u^\varepsilon$. Now since two derivatives are being taken, one loses a factor of ε^{-2} in establishing local Lipschitzness of the ODE term, but again this is irrelevant thanks to the energy estimate. Though we don't justify it here, we also remark that as $\nu \to 0$ it can be shown that the corresponding solution converges in L^2 to the Euler solution, and hence in any smaller H^s norm by interpolation.

4.5 Mild solutions for the Navier-Stokes equations and semigroups

Now we consider the question of local well-posedness in critical spaces for Navier Stokes. Rather than looking at H^s solutions, in these spaces, we will simply require that solutions satisfy the Duhamel formulation corresponding to a nonlinear heat equation.

Theorem 17. *Local well posedness in $\dot{H}^{1/2}$.*

Proof. Suppose $\nu = 1$. The point is to view the equation as a nonlinear heat equation and treat the transport term perturbatively. We will need the following estimate for the fundamental solution:

$$\left\| e^{t\Delta} \right\|_{W^{k,p} \to W^{\ell,q}} \lesssim t^{3/2(1/q-1/p)}$$

We write

$$\partial_t u - \nabla u + \mathbb{P}\nabla(u \otimes u) = 0, \nabla \cdot u = 0$$

or, equivalently,

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla(u(\tau) \otimes u(\tau))d\tau =: e^{t\Delta}u_0 - \int e^{(t-\tau)\Delta}B(u,u).$$

Now define $X_T = L^4(\dot{H}^1)$. It suffices to show the RHS of the above equation admits a fixed point in X_T. Consider the set given by $\{\rho \mid \|\rho\|_{X_T} \leq \varepsilon, \|\rho\|_{L^\infty \dot{H}^{1/2}} \leq \|u_0\|_{\dot{H}^1} + \varepsilon\}$. This is closed in X_T by lower semicontinuity of weak limits. Now observe by the standard energy estimate and dominated convergence that

$$\left\| e^{t\Delta}u_0 \right\|_{X_T} \leq \left\| e^{t\Delta}u_0 \right\|_{L^\infty \dot{H}^{1/2}} \left\| e^{t\Delta}u_0 \right\|_{L^2 \dot{H}^{3/2}} \to 0$$

as $t \to 0$, since the energy estimate tells us $e^{t\Delta}u_0 \in L^2\dot{H}^{3/2}$. We can thus fix T such that the LHS is uniformly less than ε.

We claim now that

$$\left\| \int e^{(t-\tau)\Delta}B(f,g) \right\|_{X_T} \lesssim \|f\|_{X_T} \|g\|_{X_T}.$$

To show this, we note that

$$\begin{aligned} \left\| \int e^{(t-\tau)\Delta}B(f,g) \right\|_{L^4\dot{H}^1} &\lesssim \left\| (t-\tau)^{-3/4} \|B(f,g)\|_{\dot{H}^{-1/2}} \right\|_{L^4} \\ &\lesssim \left\| (t-\tau)^{-3/4} \|f \otimes g\|_{\dot{H}^{1/2}} \right\|_{L^4} \\ &\lesssim \left\| (t-\tau)^{-3/4} \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^1} \right\|_{L^4} \end{aligned}$$

and then we can apply the Hardy-Littlewood-Sobolev inequality. □

Theorem 18. *Local well posedness in L^3.*

Proof. We use the auxiliary norm $\rho \mapsto \sup t^{1/4} \|\rho\|_{L^6}$. By scaling properties of the heat kernel, one can show the boundedness of the nonlinearity in the same way, then use the same fixed point argument. □

4.6 Yudovich theory of vorticity solutions to 2D incompressible Euler

Theorem 19. *Local well posedness with vorticity in $L^1 \cap L^\infty$.*

Proof. We first define a weak solution of the Euler equations with vorticity in $L^1 \cap L^\infty$ to be a solution satisfying

$$\int\int D_t\varphi\omega = \int_{t=T} \phi\omega - \int_{t=0} \phi\omega$$

for all test functions ϕ, where $v = K * \omega$ in the material derivative. The idea is to construct global smooth solutions $\omega^\varepsilon, v^\varepsilon$ via mollification, then to pass to an appropriate subsequential limit.

The first thing to note is that v_ε is bounded (since the kernel is nonsingular) and log Lipschitz continuous uniformly in $\|\omega\|_{L^1 \cap L^\infty}$. This allows us to use Arzela-Ascoli and pass to a subsequential limit. Showing that the limit is a weak solution then amounts to using the fact that the convergence above can be taken to be uniform on compact sets (and then using the compact support of any test function). □

5 Harmonic Analysis

5.1 Basic interpolation theorems (Marcinkiewicz, Riesz-Thorin), properties of weak L^p spaces

Theorem 20 (Riesz-Thorin). *Let T be a linear operator that is $L^{p_0} \to L^{q_0}$ bounded and $L^{p_1} \to L^{q_1}$ bounded. Then T is $L^{p_\theta} \to L^{q_\theta}$ bounded, with norm at most $\|T\|_{p_0 \to q_0}^{1-\theta} \|T\|_{p_1 \to q_1}^{\theta}$.*

Definition 21. *The weak L^p space $L^{p,\infty}$ is the space of functions f for which*

$$\sup_{\lambda > 0} \lambda \mu(\{f > \lambda\})^{1/p} < \infty$$

with norm given by the tightest constant such that the above holds.

Lemma 22. *Weak L^p spaces are complete.*

Theorem 23 (Marcinkiewicz). *Suppose T is a quasilinear operator: that is, an operator satisfying $|T(f+g)(x)| \lesssim |Tf(x)| + |Tg(x)|$ for all f, g. Suppose further that T is bounded from L^{p_i} to weak L^{q_i} for $i = 0, 1$. Then T is bounded from L^{p_θ} to L^{q_θ}.*

5.2 Calderon–Zygmund theory of singular integrals.

Definition 24. *Suppose $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ satisfies the following:*

(a) $|K(x)| \lesssim r^{-d}$

(b) $\int_{|x|>2|y|} |K(x) - K(x-y)|\, dx \lesssim 1$ *for all y.*

(c) $\int_{r \in [a,b]} K(x) dx = 0$.

Then K is called a Calderon-Zygmund kernel.

Lemma 25. *The principal value of a kernel of the type above, defined via*

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x-y) f(y) dy$$

is well-defined and L^2 bounded.

Proof. To show that the limit exists, note that the integral is equal to

$$\int_{|x-y|>\varepsilon} K(x-y)(f(y) - f(x)) dy$$

which is dominated pointwise by $C|x-y|^{-d+1}$, which is integrable near 0.
To show L^2 boundedness, it is enough to argue that the Fourier transform of a truncated version of the kernel is bounded uniformly in the truncation. Towards this, we compute that

$$\int e^{-2\pi i x \xi} K(x) \chi_{B(0,N)-B(0,1/N)} dx = \int_{|x|<|\xi|^{-1}} e^{-2\pi i x \xi} K(x) \chi dx + \int_{|x|>|\xi|^{-1}} e^{-2\pi i x \xi} K(x) \chi dx.$$

The first term is controlled by

$$\int_{1/N<|x|<|\xi|^{-1}} (e^{-2\pi i x \xi} - 1) K(x) dx \lesssim \int_{1/N<r<|\xi|^{-1}} |\xi|\, r r^{-d} (r^{d-1}) dr \lesssim 1.$$

Changing variables, we also see that the second term is equal to

$$\begin{aligned}
&\frac{1}{2}\left(\int_{N>|x|>|\xi|^{-1}} e^{-2\pi i x\xi} K(x)dx - \int_{N>|x|>|\xi|^{-1}} e^{-2\pi i (x+\xi/2|\xi|)\xi} K(x)dx\right) \\
&= \frac{1}{2}\left(\int_{N>|x|>|\xi|^{-1}} e^{-2\pi i x\xi} K(x)dx - \int_{N>\left|x-\frac{\xi}{2|\xi|^2}\right|>|\xi|^{-1}} e^{-2\pi i x\xi} K\left(x - \frac{\xi}{2|\xi|^2}\right) dx\right) \\
\le \frac{1}{2}&\left(\int_{N>|x|>|\xi|^{-1}} e^{-2\pi i x\xi}(K(x) - K(x-y))dx + \int_{\{|x-y|\in(|\xi|^{-1},N)\}\oplus\{|x|\in(|\xi|^{-1},N)\}} |x|^{-d}\, dx\right) \lesssim 1
\end{aligned}$$

where $y = \frac{\xi}{2|\xi|^2}$. □

Lemma 26 (Calderon-Zygmund decomposition). *Let $f \in L^1(\mathbb{R}^d), \lambda > 0$. Then f can be written as $f = g + b$, where $\|g\|_{L^\infty} \le \lambda$ and $b = \sum \chi_Q f$, where $Q \in B$, a collection of cubes satisfying the bounds*

$$\lambda \le ⨍_Q |f| \le 2^d \lambda, \mu(\cup B) \le \|f\|_{L^1}/\lambda.$$

Proof. Let B_i be the collection of dyadic cubes with side length 2^i. Choose i sufficiently large so that $⨍_Q |f| < \lambda$ for all $Q \in B_i$. For $x \in \mathbb{R}^d$, set $\tau(x) \in \mathbb{Z}$ to be the first i such that $⨍_Q |f| > \lambda$, where $x \in Q \in B_i$. Let B be the collection of all such Q. Then outside of $\cup B$ where $\tau = \infty$, $f \le \lambda$ almost everywhere by Lebesgue differentiation theorem. □

Lemma 27. *Operators of the form above are also bounded from L^1 to weak L^1.*

Proof. Let $\lambda > 0$ and $f = g + b$ be as in the Calderon-Zygmund decomposition. For $Q \in B$, set $\bar{f}_Q := ⨍_Q f$ to be the mean of f and $f_Q := \chi_Q(f - \bar{f}_Q)$, which is f modified to be mean 0 and supported on Q. Observe that

$$f = (g + \sum_Q \chi_Q \bar{f}_Q) + \sum_Q f_Q =: f_1 + f_2.$$

By construction, $|f_1| \lesssim \lambda, \|f_1\|_{L^1} \lesssim \|f\|_{L^1}$, and also $\|f_2\|_{L^1} \lesssim \|f\|_{L^1}$. Hence

$$|\{Tf > \lambda\}| \le |\{Tf_1 > \lambda/2\}| + |\{Tf_2 > \lambda/2\}|.$$

The first term is estimated by

$$\|Tf_1\|_{L^2}^2/\lambda^2 \lesssim \|f_1\|_{L^2}^2/\lambda^2 \lesssim \lambda\|f\|_{L^1}/\lambda^2 = \|f\|_{L^1}/\lambda$$

which is admissible. To control the second term, recall that $|\mathrm{supp}\, f_2| \le \|f_1\|/\lambda$, so it suffices to estimate

$$|\{Tf_2(x) > \lambda \mid x \notin C\,\mathrm{supp}\, f_2\}|$$

where $C\,\mathrm{supp}\, f_2$ is obtained by taking scaling each $Q \in B$ by a factor C and then taking the union. To do this, we again use an L^1 type bound:

$$|\{Tf_2(x) > \lambda/2 \mid x \notin C\,\mathrm{supp}\, f_2\}| \le \frac{2}{\lambda}\int_{(C\,\mathrm{supp}\, f_2)^c} |Tf_2| \le \frac{2}{\lambda}\sum_Q \int_{(C\,\mathrm{supp}\, f_2)^c} |Tf_Q|.$$

Now since each f_Q is mean 0, for x away from $\mathrm{supp}\, f_2$,

$$Tf_Q(x) = \int K(x-y)f_Q(y)dy = \int (K(x-y) - K(x-y_Q))f_Q(y)dy$$

so integrating this gives

$$\begin{aligned}
\int_{(C\,\mathrm{supp}\, f_2)^c} |Tf_Q| &= \int_{(C\,\mathrm{supp}\, f_2)^c} \int_Q |(K(x-y) - K(x-y_Q))|\, |f_Q(y)|\, dydx \\
&= \int_Q \int_{(C\,\mathrm{supp}\, f_2)^c} |(K(x-y) - K(x-y_Q))|\, |f_Q(y)|\, dxdy \\
&\lesssim \int |f_Q(y)|\, dxdy
\end{aligned}$$

where in the last line we use the fact that $|x - y_Q| > C \operatorname{diam} Q > C(y - y_Q)$ so that the Hormander condition applies with $x = x - y_Q$ and $y = y - y_Q$. Summing over all Q gives $\|f\|_{L^1}$ as desired. ☐

Lemma 28. *T is also Holder bounded for functions with compact support.*

Proof. Let $f \in C^\alpha(\mathbb{R}^d)$ be supported inside the unit ball. Then for all $|x| \leq 2$,

$$|Tf(x)| = \left| \int K(x-y) f(y) \right| = \left| \int K(x-y) f(y) - f(x) \right| \lesssim \int_{|x-y|\leq 3} |x-y|^{-d} |x-y|^\alpha \lesssim 1$$

and for all $|x| \geq 3$, $K \lesssim 1$ on the support of the convolution, so $|Tf(x)| \lesssim \|f\|_{L^1} \lesssim 1$.
Now let $\delta := |x - x'| < 1$. Then

$$\begin{aligned}
Tf(x) - Tf(x') &= \int K(y)(f(x-y) - f(x'-y))dy \\
&= \int_{|y|<3\delta} K(y)(f(x-y) - f(x) - f(x'-y) + f(x'))dy + \int_{|y|>3\delta} K(y)(f(x-y) - f(x'-y))dy
\end{aligned}$$

and the first term is estimated by

$$\int_{|y|<3\delta} |y|^{-d+\alpha} \lesssim \delta^\alpha.$$

To estimate the second term, we let $K_\delta := \chi_{|x|>3\delta} K$ be the truncated version of the kernel, and change variables in the convolution to obtain

$$\int (K_\delta(x-y) - K_\delta(x'-y)) f(y) = \int (K_\delta(x-y) - K_\delta(x'-y))(f(y) - f(x)).$$

Note also that the first term is supported within $|x - y| \geq 2\delta$, since $|x - x'| = \delta$. Furthermore, within this range, by the mean value theorem, we have

$$K_\delta(x-y) - K_\delta(x'-y) = (x - x') \cdot \nabla K(x^* - y)$$

for some x^* in the segment joining x, x'. Now by the triangle inequality, we know that $|x^* - y|$ differs by $|x - y|$ by at most δ, and hence is absolutely comparable to $|x - y|$ in this region. We deduce that the second term is controlled by

$$\int_{|x-y|>2\delta} \delta\, |x-y|^{-d-1} |x-y|^\alpha \lesssim \delta^\alpha$$

as desired. ☐

Now we go through some examples.

Lemma 29. *Consider the Newton potentials, given by the kernels*

$$K(x) = \begin{cases} C_d |x|^{2-d} & d \geq 3 \\ C_d \log |x| & d = 2. \end{cases}$$

Convolution with these kernels defines an inverse to Δ; that is, $\Delta(K * f) = f$ for all decaying f.

Proof. We focus on the case $d \geq 3$; the calculation for $d = 2$ is similar. First we observe that

$$\nabla K = x\,|x|^{-d}.$$

Now

$$\Delta K(x) = \int K(x-y)\Delta f(y)dy = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x-y)\Delta f(y)dy.$$

For each ε we integrate by parts, so that the right hand side is equal to

$$-\int_{|x-y|>\varepsilon} \nabla f(y) \cdot \nabla K + \int_{|x-y|=\varepsilon} |x-y|^{2-d}\, \partial_r f r^{-d+1} d\sigma$$

where $d\sigma$ is the volume form on the unit sphere. The second term is $\mathcal{O}(\varepsilon)$. The first term may be rewritten as

$$-\int_{\varepsilon}^{\infty} \partial_r f(x + r\xi) r^{-d+1} r^{d-1} d\sigma \to f(x)$$

as desired. □

Lemma 30. *With Δ^{-1} defined as above, $\partial^2_{ij}\Delta^{-1}$ is a singular integral operator.*

Proof. Use Mikhlin multiplier theorem. □

5.3 Littlewood–Paley theory

Definition 31. *Let $m : \mathbb{R}^d \to \mathbb{C}$ be a bounded function. The associated multiplier operator T_m is given by*

$$T_m f = \int e^{2\pi i x\xi} m(\xi)\hat{f}(\xi)d\xi$$

and is L^2 bounded by Plancherel, with $\|T_m\|_{2\to 2} = \|m\|_{\infty}$.

Lemma 32. *With m as above, if $|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}$ for all $|\alpha| \leq d+2$, then T_m is L^p bounded for all $p \in (1, \infty)$.*

Proof. By Calderon-Zygmund theory, it suffices to show that

$$K := \sum_{|i|\leq N} m_i^{\vee}$$

decays like r^{-d} (and it's gradient decays one power better) uniformly in all sufficiently large N, where m_i is the multiplier localized to frequencies $\in [2^{-i}, 2^i]$. Firstly, we have

$$|\partial^\alpha m_i(\xi)| \lesssim |\xi|^{-|\alpha|} \lesssim 2^{-i|\alpha|}, |\xi_\mu \partial^\alpha m_i(\xi)| \lesssim |\xi|^{-|\alpha|} \lesssim 2^{-i(|\alpha|+1)}$$

where in the last bounds we pass to one uniform over the support of m_i. Integrating this gives

$$\|\partial^\alpha m_i(\xi)\|_{L^1} \lesssim 2^{i(d-|\alpha|)}, \|\xi_\mu \partial^\alpha m_i(\xi)\|_{L^1} \lesssim 2^{i(d+1-|\alpha|)}.$$

Now fix x, set $\alpha = 0$, take the inverse Fourier transform, and sum over all $\mu, 2^i \leq |x|^{-1}$ to find

$$\sum_{2^i \leq |x|^{-1}} m_i^{\vee}(x) \leq \sum_{2^i \leq |x|^{-1}} 2^{id} \lesssim |x|^{-d}, \sum_{2^i \leq \nabla |x|^{-1}} m_i^{\vee}(x) \leq \sum_{2^i \leq |x|^{-1}} 2^{id+i} \lesssim |x|^{-d-1}.$$

On the other hand, taking the inverse Fourier transform and summing over all $|\alpha| = d+2$ yields

$$|m_i^{\vee}(x)| \lesssim 2^{-2i}\,|x|^{-d-2}, \nabla m_i^{\vee}(x) \lesssim 2^{-i}\,|x|^{-d-2}$$

so summing over all $2^i \geq |\xi|^{-1}$ yields the desired estimate. □

We now take a detour to review some probabilistic items that will be of use.

Lemma 33. *Let $\{a_i\}_{i=1}^N$ be such that $\|a_i\|_{\ell^2} = 1$ and $\{r_i\}$ be a sequence of iid random signs. Then $S_N := \sum a_i r_i$ has sub-Gaussian tails, e.g.*

$$\mathbb{P}(S_N > \lambda) \leq e^{-\lambda^2/2}.$$

Proof. For $t > 0$ to be fixed later, we have

$$\mathbb{E}e^{tS_N} = \prod \mathbb{E}e^{ta_i r_i} = \prod \cosh(ta_i r_i) \leq \prod e^{(ta_i r_i)^2/2} = e^{t^2/2}.$$

Now by Chebyshev,

$$\mathbb{P}(S_N > \lambda) = \mathbb{P}(e^{tS_N} > e^{t\lambda}) \leq e^{t^2/2}/e^{t\lambda} \leq e^{-\lambda^2/2}$$

where the last inequality follows by choosing $t = \lambda$. □

Theorem 34 (Khinchin). *Let $\{r_i\}$ be a sequence of iid random signs. For any $p \in [1, \infty)$, we have*

$$\|a\|_{\ell^2} \lesssim \left(\mathbb{E}\left|\sum r_i a_i\right|^p\right)^{1/p} \lesssim \|a\|_{\ell^2}$$

uniformly in the length of the sequence a.

Proof. We first show the upper bound. Suppose $\|a\|_{\ell^2} = 1$ and set $S_N := \sum r_i a_i$. Then by the previous lemma,

$$\mathbb{E}\left|\sum r_i a_i\right|^p = \int \mathbb{P}(|S_N| > \lambda) p\lambda^{p-1} d\lambda \lesssim \int e^{-\lambda^2/2} p\lambda^{p-1} d\lambda \lesssim 1.$$

Now to show the lower bound, observe by Jensen that

$$\left(\mathbb{E}\left|\sum r_i a_i\right|\right) \leq \left(\mathbb{E}\left|\sum r_i a_i\right|^p\right)^{1/p}$$

so it suffices to handle the case $p = 1$. Since $\mathbb{E}S_N^2 = \mathbb{E}\sum (r_i a_i)^2 = \|a\|_{\ell^2}$, we then have

$$\|a\|_{\ell^2} = \mathbb{E}S_N^2 = \mathbb{E}\,|S_N|^{2/3}\,|S_N|^{4/3} \leq (\mathbb{E}\,|S_N|)^{2/3}(|S_N|^4)^{1/3} \lesssim (\mathbb{E}\,|S_N|)^{2/3}\,\|a\|_{\ell^2}$$

which is the result. □

Theorem 35 (Littlewood-Paley square function). *Let P_i be the projection onto frequencies $\sim 2^i$ with a smooth cutoff ψ. Define the square function $Sf(x) := \|P_i(x)\|_{\ell^2}$. Then for all $p \in (1, \infty)$,*

$$\|f\|_{L^p} \asymp \|Sf\|_{L^p}.$$

Proof. Let $\{r_i\}$ be a sequence of iid random signs. Define the random Fourier multiplier $m_N := \sum_{|i| \leq N} r_i \psi_i$. We claim that the corresponding kernel satisfies the hypotheses of the Mikhlin multiplier theorem uniformly in N. Indeed, by scaling, each ψ_i satisfies $\partial^\alpha \psi(\xi) \lesssim |\xi|^{-|\alpha|}$ on the support of ψ_i, and only finitely many ψ_i are nonzero at each ξ, so we conclude. Now using the previous result, we have

$$\int |Sf(x)|^p = \lim_{N\to\infty} \int \left|\sum_{|i|\leq N} P_i f\right|^{p/2} \leq \limsup_N \int \mathbb{E}\left(\sum_{|i|\leq N} r_i P_i f\right)^p \leq \limsup_N \mathbb{E}\int (m_N f)^p \lesssim \|f\|_{L^p}^p.$$

For the lower bound, we can use duality. Let $\bar{\psi}_j$ be a bump function adapted in the same way to scale 2^j such that $\bar{\psi}_j = 1$ on the support of ψ_j. Then

$$\begin{aligned}\|f\|_{L^p} &= \sup_{\substack{\|g\|_{L^{p'}}=1 \\ g\in\mathcal{S}}} \langle f, g\rangle = \sup_g \left\langle \hat{f}, \hat{g}\right\rangle \\ &= \sup_g \sum_i \left\langle \psi_j \hat{f}, \hat{g}\right\rangle = \sup_g \sum_i \left\langle \psi_j \hat{f}, \bar{\psi}_j \hat{g}\right\rangle \leq \|Sf\|_{L^p}\,\|Sg\|_{L^q} \lesssim \|Sf\|_{L^p}.\end{aligned}$$

□

5.4 Almost orthogonality

Theorem 36 (Cotlar's lemma)**.** *Let V be a Hilbert space and $T_n : V \to V$ be a sequence of operators satisfying the bound*

$$\|T_i^* T_j\|, \|T_i T_j^*\| \le (\gamma(i-j))^2.$$

Then $\|\sum T_i\| \le \|\gamma\|_{\ell^1}$.

Proof. Let $T := \sum T_i$ and $B := \sup \|T_n\|$. Consider expanding the product $(T^*T)^n$. We may bound a typical term by

$$\left\|T_{i_1}^* T_{j_1}^* \cdots T_{i_n}^* T_{j_n}\right\| \le \|T_{i_1}\| \|T_{j_n}\| \prod \left\|T_{j_k} T_{i_{k+1}}^*\right\| \le B^2 (\prod \gamma(j_k - i_{k+1}))^2$$

or, by reassociating,

$$\left\|T_{i_1}^* T_{j_1}^* \cdots T_{i_n}^* T_{j_n}\right\| \le \prod \left\|T_{j_k} T_{i_{k+1}}^*\right\| \le (\prod \gamma(i_k - j_k)).$$

Hence

$$\|(T^*T)^n\| \le \sum_{i_1, \cdots, i_n, j_1, \cdots, j_n} B\gamma(i_1 - j_1)\gamma(j_1 - i_2) \cdots \gamma(j_{n-1} - i_n) \le NB \|\gamma\|_{\ell^1}^{2n-1}.$$

Now by the spectral theorem, $\|T\| = \lim_{n\to\infty} \|(T^*T)^n\|^{1/2n}$, so we conclude. □

Theorem 37 (Schur's lemma)**.** *Consider an operator of the form*

$$Tf(x) = \int K(x,y) f(y) d\mu(y).$$

Then

(a) $\|T\|_{1\to 1} \le \sup_y \|K(\cdot, y)\|_1$.

(b) $\|T\|_{\infty\to\infty} \le \sup_x \|K(x, \cdot)\|_1$.

(c) $\|T\|_{1\to\infty} \le \|K\|_\infty$.

Theorem 38 (Calderon-Vaillancourt)**.** *Consider a ψdo of the form*

$$Tf(x) = e^{ix\xi} a(x,\xi) \hat{f}(\xi) d\xi$$

where a has $2d+1$ derivatives in both variables uniformly bounded. Then T is L^2 bounded.

Proof. Let χ be a function such that it's integer shifts form a partition of unity. Let $a_{k,\ell}$ be a truncated to a neighborhood of k in physical space and ℓ in frequency space, and define

$$T_{k,\ell} f(x) = \int e^{ix\xi} a_{k,\ell} f(\xi) d\xi.$$

It suffices to show that $\left\|T_{k,\ell} T_{k+\alpha,\ell+\beta}^*\right\| \lesssim \alpha^{-2d-1} \beta^{-2d-1}$ (and the version with the adjoint) since the square roots of the right hand side are summable.
To show this, note first that $T_{k,\ell}^* T_{k',\ell'}$ involves an integral of the form $a_{k,\ell}(x,\xi)\overline{a_{k',\ell'}(x,\eta)}$ and is hence 0 unless $|k-k'|$ is small, so it suffices to show decay in $|\ell - \ell'|$.
The kernel associated to this composition is given by

$$\int e^{-ix(\xi-\eta)} a_{k,\ell}(x,\xi) \overline{a_{k',\ell'}(x,\eta)}$$

and repeatedly integrating this by parts gives decay in $|\xi - \eta| \sim |\ell - \ell'|$. □

Theorem 39 (Hardy)**.** *Suppose* $0 \le s < d/2$*. Then*

$$\left\| |x|^{-s} f \right\|_{L^2} \lesssim \|f\|_{\dot{H}^s}.$$

Proof. We first prove the estimate assuming f is localized in Fourier space. By scaling, it suffices to prove the estimate at scale 0, for which it further suffices to show that

$$Tf(x) = |x|^{-s} \tilde{P}_0$$

is L^2 bounded, where $\tilde{P}_0$ is a fattened Littlewood-Paley operator. Now we recall Bernstein's inequality, which says that

$$\|f\|_{L^q} \lesssim R^{d(1/p-1/q)}.$$

Now we have that

$$\|Tf\|_{L^2} \le \left\|\chi_{\{|x|<1\}} Tf\right\|_{L^2} + \left\|\chi_{\{|x|>1\}} Tf\right\|_{L^2}.$$

The second term is obviously controlled by $\|f\|_{L^2}$. The first term is controlled by

$$\sum_{i \le 0} 2^{-is} \left\|\chi_{\{|x|\sim 2^i\}} f\right\|_{L^2} \lesssim \sum_{i\le 0} 2^{-is} 2^{id/2} \|f\|_{L^\infty} \lesssim \|f\|_{L^2}$$

where the last inequality follows by Bernstein or Sobolev embedding.
Now we consider the estimate assuming f is localized in physical space. We want to prove the estimate

$$\|\psi f\|_{L^2} \lesssim \|f\|_{\dot{H}^s}$$

for ψ a cutoff supported near $|x| = 1$. We note that

$$\begin{aligned}
\|\psi f\|_{L^2}^2 &\lesssim \sum_{k\ge 0} \|\psi P_k f\|_{L^2}^2 + \left(\sum_{k<0} \|\psi P_k f\|_{L^2}\right)^2 \\
&\lesssim \sum_{k\ge 0} 2^{2sk} \|P_k f\|_{L^2}^2 + \left(\sum_{k<0} \|P_k f\|_{L^\infty}\right)^2 \\
&\lesssim \|f\|_{\dot{H}^s}^2 + \left(\sum_{k<0} 2^{kd/2} \|P_k f\|_{L^2}\right)^2 \\
&\lesssim \|f\|_{\dot{H}^s}^2 + \left(\sum_{k<0} 2^{k(d/2-s)}\right) \left(\sum 2^{2sk} \|P_k f\|_{L^2}^2\right) \lesssim \|f\|_{\dot{H}^s}^2.
\end{aligned}$$

Now to proceed to the final result, we use that

$$\begin{aligned}
\left\| |x|^{-s} f \right\|_{L^2}^2 &\lesssim \sum_{\ell} 2^{-2\ell s} \|\psi_\ell f\|_{L^2}^2 \\
&\lesssim \sum_{\ell} \left(\sum_{k+\ell\le 0} \|\psi_\ell P_k f\|_{L^2}\right)^2 + \sum_{\ell} 2^{-2\ell s} \|\psi_\ell P_{>-\ell} f\|_{L^2}^2.
\end{aligned}$$

We handle each of these terms in turn. The first term is controlled by

$$\sum_{\ell} \left(\sum_{k+\ell\le 0} 2^{(d/2-s)(\ell+k)} 2^{sk} \|P_k f\|_{L^2}^2\right)^2.$$

View this as mapping the function $k \mapsto 2^{sk} \|P_k f\|_{L^2}^2$ with the kernel

$$K(k, \ell) = 2^{(d/2-s)(\ell+k)} \mathbf{1}_{\{k+\ell \le 0\}}.$$

This is $L^\infty L^1$ bounded in both directions, so we conclude. For the second, we discard the spatial localization, so that

$$\begin{aligned}\sum_\ell 2^{-2\ell s} \|\psi_\ell P_{>-\ell} f\|_{L^2}^2 &\le \sum_\ell 2^{-2\ell s} \|P_{>-\ell} f\|_{L^2}^2 \\ &\le \sum_\ell \sum_{k>-\ell} 2^{-2\ell s - 2sk} 2^{2sk} \|P_k f\|_{L^2}^2 \le \sum_k 2^{2sk} \|P_k f\|_{L^2}^2 \sum_{\ell > -k} 2^{-2\ell s - 2sk}.\end{aligned}$$

$\square$